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A RESISTOR-NETWORK SOLUTION FOR THE ELASTO-PLASTIC TORSION PROBLEM

J. H. Weiner,[1] M. G. Salvadori,[2] M. ASCE, and V. Paschkis[3]

INTRODUCTION

The solution of torsion problems in which the stresses exceed the elastic limit presents great analytical difficulties. Numerical solutions, using relaxation techniques, have been developed[4] and are effective. However, if many mesh points and separate calculations are required, as for example in obtaining an accurate torque vs. angle of twist relationship, the relaxation technique becomes quite laborious.

In this paper a method of solution using a resistor network is presented for the torsion of a cylinder of perfectly elasto-plastic material with convex, simply-connected cross section. The finite-difference approximations customarily used in the relaxation procedure are made, but the relaxation is accomplished automatically by the network. Thus, once the network for a given geometry is set up, a complete torque vs. twist curve may be obtained in a short time.

Electrical Analog of the Elasto-plastic Torsion Function

The method will be described as applied to the torsion of a square section of side a. The stress-strain relationship of the material is assumed to be as shown in Fig. 1.

In the usual notation, the components of shear stress on the cross section are given by:[5]

$$\tau_{zx} = \frac{\partial \phi}{\partial y}; \quad \tau_{zy} = -\frac{\partial \phi}{\partial x} \tag{1}$$

where the stress function $\phi(x,y)$ satisfies the equation:

$$\Delta^2 \phi + 2G\theta = 0 \tag{2}$$

in the elastic portion of the cylinder and the over-riding conditions

$$\left|\Delta \phi\right| = \tau_o; \tag{3a}$$

$$\Delta^2 \phi + 2G\theta \geq 0 \tag{3b}$$

in the plastic portion. Equation (3a) corresponds, in terms of the membrane

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4. R. S. Southwell, "Relaxation Methods in Mathematical Physics," Oxford University Press, Ch. 6.
5. M. Hetenyi, "Handbook of Experimental Stress Analysis," Ch. 16.

analogy, to the erection of a roof of slope τ_o confining the membrane. Equation (3b) corresponds to the restriction that the roof may only push, not pull, on the membrane. In addition ϕ satisfies on the boundary C the condition:

$$\phi_C = 0 \tag{4}$$

The torque, M, corresponding to a given $\phi(x,y)$, is given by:

$$M = 2 \int_A \int \phi \, dx \, dy \tag{5}$$

For the study of elasto-plastic torsion it is convenient to introduce the following dimensionless quantities:

$$\xi = \frac{x}{a} \; ; \quad \eta = \frac{y}{a} \; ; \quad \Psi = \frac{\phi}{Ga} \tag{6}$$

Then equations (1) to (5) become,

$$\tau_{zx} = G \frac{\partial \Psi}{\partial \eta} \text{ and } \tau_{zy} = - G \frac{\partial \Psi}{\partial \xi} \tag{1'}$$

$$\Delta^2 \Psi + 2\theta a = 0 \tag{2'}$$

$$\left| \Delta \Psi \right| = \frac{\tau o}{G} \; ; \tag{3'a}$$

$$\Delta^2 \Psi + 2\theta a \geqslant 0 \tag{3'b}$$

$$\Psi_C = 0 \tag{4'}$$

$$M = 2Ga^3 \int_A \int \Psi \, d\xi \, d\eta = 2G^3 \, V \tag{5'}$$

Introducing the usual finite difference approximations, with mesh size $h = \frac{1}{2n}$, Fig. 2, equation (2') becomes:

$$\frac{\Psi_r + \Psi_\ell + \Psi_a + \Psi_b - 4\Psi_i}{h^2} + 2\theta a = 0 \tag{2''}$$

In finite difference terms, equation (3'a) becomes, using equation (4'):

$$\Psi_m \leqslant \frac{mh\tau o}{G}, \quad m = 1, 2, \ldots, n \tag{3''a}$$

where Ψ_m is the value of Ψ at any node a normal distance mh from the boundary. Similarly equation (3'b) becomes:

$$\frac{\Psi_r + \Psi_\ell + \Psi_a + \Psi_b - 4\Psi_i}{h^2} + 2\theta a \geqslant 0 \tag{3''b}$$

for any node i in the plastic region.

Consider now the resistor network shown in Fig. 3. Applying Kirchhoff's law to the currents flowing into a typical node, the following relation is obtained:

$$\frac{e_r - e_i}{R} + \frac{e_\ell - e_i}{R} + \frac{e_a - e_i}{R} + \frac{e_b - e_i}{R} + \frac{E - e_i}{R_N} = 0$$

If E is made very large relative to any e_i, the last term is approximately equal to E/R_N, and the above equation becomes:

$$\frac{e_r + e_\ell + e_a + e_b - 4e_i}{R} + \frac{E}{R_N} = 0 \qquad (2''')$$

If a scale factor between e and Ψ is established

$$e = K\Psi, \qquad (7)$$

equations (2'') and (2''') become identical for

$$K = \frac{ER}{2R_N\theta\ ah^2} \qquad (8)$$

The analogs of equations $(3''_{a,b})$ then become:

$$\text{(a)}\ e_m \leq Kmh\ \tau_o/G \qquad (3'''a)$$

$$\text{(b)}\ \frac{e_r + e_\ell + e_a + e_b - 4e_i}{R} + \frac{E}{R_N} \geq 0, \qquad (3'''b)$$

where e_m is the value of e at any node a normal distance mh from the boundary.

The boundary condition on Ψ, equation (4'), is satisfied by shorting the boundary nodes together and connecting them to ground, i.e., the zero of the voltage scale.

In order to satisfy equation (3''' a), a low internal impedance voltage source is required with taps at $mKh\ \tau_o/G$ (m = 1,2, . . . , n). In practice, K is chosen so that $Kh\tau_o/G = e_s$, the voltage across a single cell of a storage battery. Starting at the node closest to the centroid of the section, a normal distance h from the boundary, since this is the node with highest slope, the voltage is compared with e_s by means of a difference amplifier. If the nodal voltage is greater than e_s, it is shorted to e_s; if it is lower than e_s it remains untouched. Then the next nodes in the same row are compared with e_s in an identical way. Following examination of the nodes in the first row, the nodes in the second row a normal distance 2h from the boundary are compared with $2e_s$, etc.

It is seen that equation (3''' b) is automatically satisfied for any node completely surrounded by nodes in the plastic region. A numerical check, made by substituting the appropriate voltages in equation (3'''b) is required for the nodes, on the border between the elastic and the plastic regions, and those at which it is not satisfied are disconnected from the battery. However, if the nodes are connected to the battery in the order described above, this reversal will seldom occur.

In this manner, there is established a complete analogy between the finite difference equations governing Ψ and those governing the nodal voltages in the electrical circuit. To determine the values of Ψ, therefore, it is only necessary to measure the nodal voltages and use equation (7) to get the equivalent values of Ψ.

To determine the torque, M, corresponding to these nodal values of Ψ, it is convenient to use the trapezoidal rule for numerical integration of V equation (5'):

$$M = 2Ga^3V = 2Ga^3h^2 \sum_{i=1}^{(2n-1)^2} \Psi i = \frac{2Ga^3h^2}{K} \sum_{i=1}^{(2n-1)^2} e_i, \qquad (5'')$$

where the summation is carried out over all the internal nodal values. It should be noted that it is theoretically possible to determine the torque with a single current-difference measurement. For the current I flowing into the circuit from the voltage source E for a given set of voltages e_i is:

$$I = \frac{1}{R_N} \sum_{i=1}^{(2n-1)^2} (E - e_i) = I_o - \frac{1}{R_N} \sum_{i=1}^{(2n-1)^2} e_i,$$

where I_o is the current which would flow into the circuit if all the e_i were zero. Therefore:

$$M = \frac{2G\ R_N a^3 h^2}{K} (I_o - I) = \frac{4h^4 R_N^2}{ER} (I_o - I)\ G\theta a^4 \qquad (5''')$$

However, since $I_o - I$ is very small relative to I, its measurement is delicate and further work is required for the development of a technique giving sufficiently accurate results. The values of M in this paper, were obtained by means of equation (5'').

Determination of Torque vs. Twist Curves

It is most convenient to present a complete torque vs. twixt curve in terms of the dimensionless quantities M/M_o and θ/θ_o where M_o and θ_o are, respectively, the torque and angle of twist at which plastic deformation just begins, i.e., at which the maximum stress in the cross-section is τ_o.

The dimensionless slope of the roof is therefore first determined from an "elastic" network experiment, that is one in which the over-riding condition, equation (3''), is not applied. For this experiment, any convenient value of K may be used. The maximum value of $|\nabla\Psi|$, occurring at the boundary point nearest the centroid of the section, may be computed from the nodal values of Ψ by the use of suitable finite-difference formulae.

The above experiment yields the values of M_o, θ_o and the value of $\tau_o/G = |\nabla\Psi|_{max}$. The value of K is now chosen so that $Kh\,\tau_o/G = e_s$, the voltage across the single cell of a storage battery, and this choice of K is seen from equation (8) to be equivalent to a choice of E_o corresponding to the above value of θ_o. From equation (8) it is further seen that if the scale factor K is to be maintained constant, the imposition of an angle of twist $\theta = m\,\theta_o (m>1)$, corresponds to increasing E to mE_o.

The technique described above is then applied for $E = mE_o$ to obtain the torque M corresponding to an angle of twist $m\theta_o$.

Experimental Results

The technique explained in the previous sections was used to determine the torque vs. twist curve for a cylinder of square cross-section on the Heat and Mass Flow Analyzer at Columbia University School of Engineering. This equipment is suitable for circuits with as many as 576 nodes.[6,7] The three curves of Fig. 4 give M/M_o vs. θ/θ_o for h = 1/6 and h = 1/12 with a roof

6. V. Paschkis and H. D. Baker, "A Method for Determining Unsteady State Heat Transfer," Trans. ASME, Vol. 67, (1942) p. 105.
7. V. Paschkis and T. Mirsepassi, "Temperature Distribution in the Hearths of Blast Furnaces," Iron and Steel Engineer, 1954, pp. 53-66.

slope λ approximated from the nodal voltages of the "elastic" network experiment by the first difference expression at the boundary:

$$\lambda' = \frac{\tau_o}{G} = \frac{(\Delta e)_B}{Kh},$$

and for h = 1/12 with the roof slope approximated by:

$$\lambda'' = \frac{\tau_o}{G} = \frac{1}{h}\left(\Delta e - \frac{1}{2}\Delta^2 e + \frac{1}{3}\Delta^3 e - - -\right)_B \qquad (9)$$

The differences higher than the second of the "elastic" voltages e become erratic in the experiments and the best slope for the accuracy used in these measurements is obtained by the first two terms of equation (9). It will be noticed that the results are very sensitive to the value of λ, while they are not substantially influenced by lumping.

Error Evaluation

In order to check the accuracy of the experimental results some of these were checked against results obtained by relaxation.

Table I gives the experimental and relaxation values of $\Psi\theta_o a$ for θ/θ_o = 1.5 and h = 1/6 with the slope λ'. The largest and average errors equal 1.2% and 0.87%, respectively, and are to be compared with the 1% error in the voltage measurements. It is thus assumed that the other experimental results are also affected by errors of the order of 1%.

The value of the double integral V in equation (5") for θ/θ_o = 1.5 computed by the trapezoidal rule from the experimental values has an error of only 0.05% relative to the relaxation values. The same error in the value of the integral for the elastic case equals 0.33%. The experimental value of the ratio M/M_o, which depends on the elastic case integral, has an error of - 0.30% measured against the relaxation results.

The purely numerical techniques used to evaluate the pivotal values of Ψ and M introduce three sources of errors in the results: a) the approximate evaluation of the pivotal values of Ψ due to lumping, i. e., due to the spacing h; b) the approximate evaluation of V, equation (5"), i. e., the volume under the analog membrane; c) the approximate evaluation of the boundary slope λ.

To evaluate the influence of lumping it may be noticed that the error in M is of the so-called h^2-type and an h^2-extrapolation may be used from the values for h = 1/6 and h = 1/12. For the elastic case, the extrapolation gives:

$$M_o = \left[\frac{4}{3}\,0.1377 - \frac{1}{3}\,0.1291\right] G\theta_o a^4 = 0.1406\; G\theta_o a^4,$$

which checks up to the last figure with the result given in Timoshenko[8] and indicates an error of 8.18% in the h = 1/6 approximation and of 2.06% in the h = 1/12 approximation. For the elasto-plastic case θ/θ_o = 1.5 the extrapolation would not be valid since the h = 1/6 and h = 1/12 cases are evaluated by means of different roof slopes, but it may be presumed that the same order of magnitude of error as in the elastic case will be present here, i.e., that M be in error by approximately 2%.

In order to evaluate the influence of the integration procedure, V in equation (5") was computed by the trapezoidal rule and by Simpson's 1/3 rule on

8. S. Timoshenko, "Theory of Elasticity," McGraw Hill Book Co., New York, 1934, p. 249.

the basis of the relaxation results (h = 1/6) for the elastic case and for the case $\theta/\theta_o = 1.5$ with the slope λ'. It was found that the difference between the two evaluations of the integral equals 6.20% in the elastic case, and 4.98% in the case $\theta/\theta_o = 1.5$. On the other hand, the ratios of the integrals V (which is identical with the ratio M/M_o) evaluated by the two methods, has a difference of only 1.3% for $\theta/\theta_o = 1.5$ when the slope is λ' and of 0.71% when the slope is λ''.

Since the error in Simpson's rule is of order h^4, it may be concluded that the error in the ratio M/M_o due to integration by the trapezoidal rule, which is of order h^2, for h = 1/6 is less than 1%.

To compute the error due to the roof slope, the integral V of equation (5") was evaluated from the relaxation results for h = 1/6 and $\theta/\theta_o = 1.5$ with the two values λ' and λ'' of the slope, using the trapezoidal rule. The error in the λ' solution, assuming the λ'' solution to be correct, equals 17.18% and is identical with the error in M/M_o. The same error using Simpson's rule equals 17.66%.

It is thus seen that the errors due to other causes are negligible compared to the errors due to the approximation of the roof slope and that this particular parameter should be evaluated as accurately as possible whatever the method of solution adopted.

ACKNOWLEDGMENTS

The resistor-network experiments were performed by Mr. Taghi Mirsepassi and Mr. Paul Calcaterra of the Heat and Mass Flow Analyzer Laboratory, Columbia University.

Table I

Node	$(\Psi/\theta_o a) \times 10^4$		% Error
	Network	Computed	
1	1776	1793	-0.95
2	1433	1428	+0.35
3	1583	1585	-0.13
4	641	634	+1.1
5	865	855	+1.2
6	865	855	+1.2

Nodal values of $\Psi\theta_o a$
h = 1/6, Slope λ', $\theta/\theta_o = 1.5$
See Figure 2 for location of nodes.

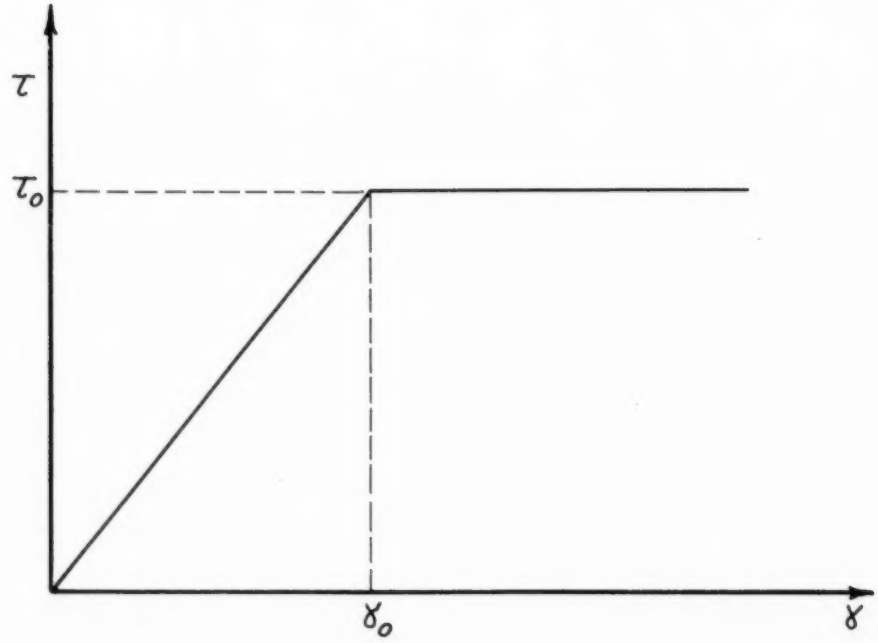

Fig. 1 Elasto-Plastic Stress - Strain Relationship

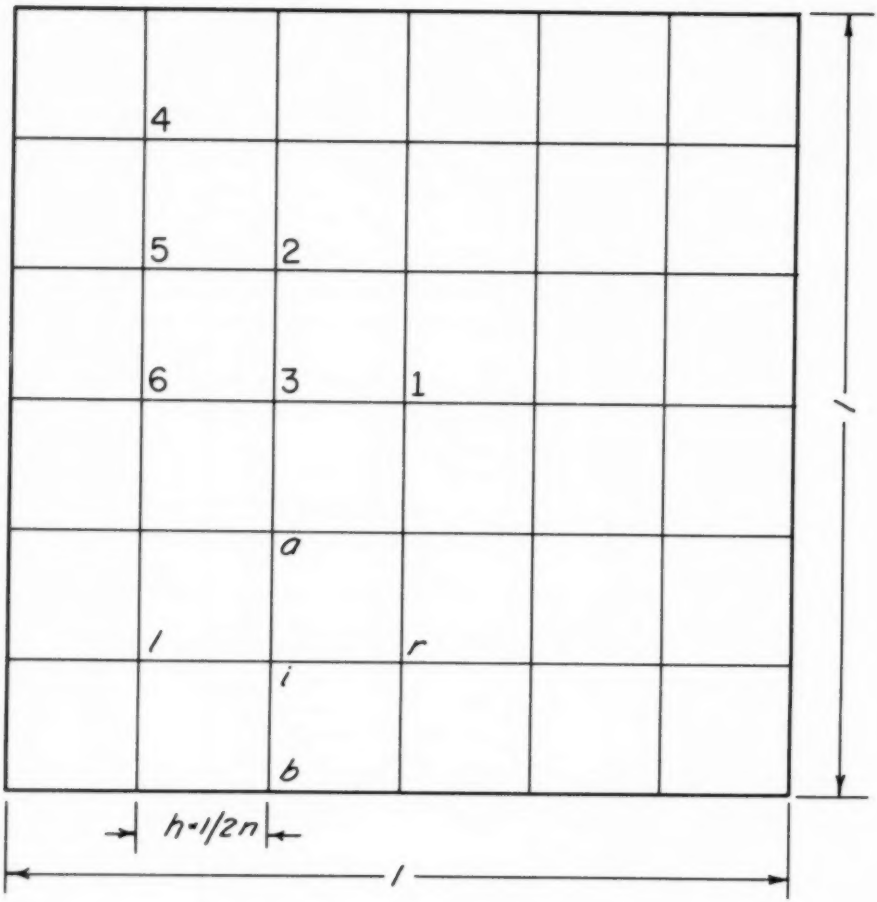

Fig. 2 Finite Difference Mesh

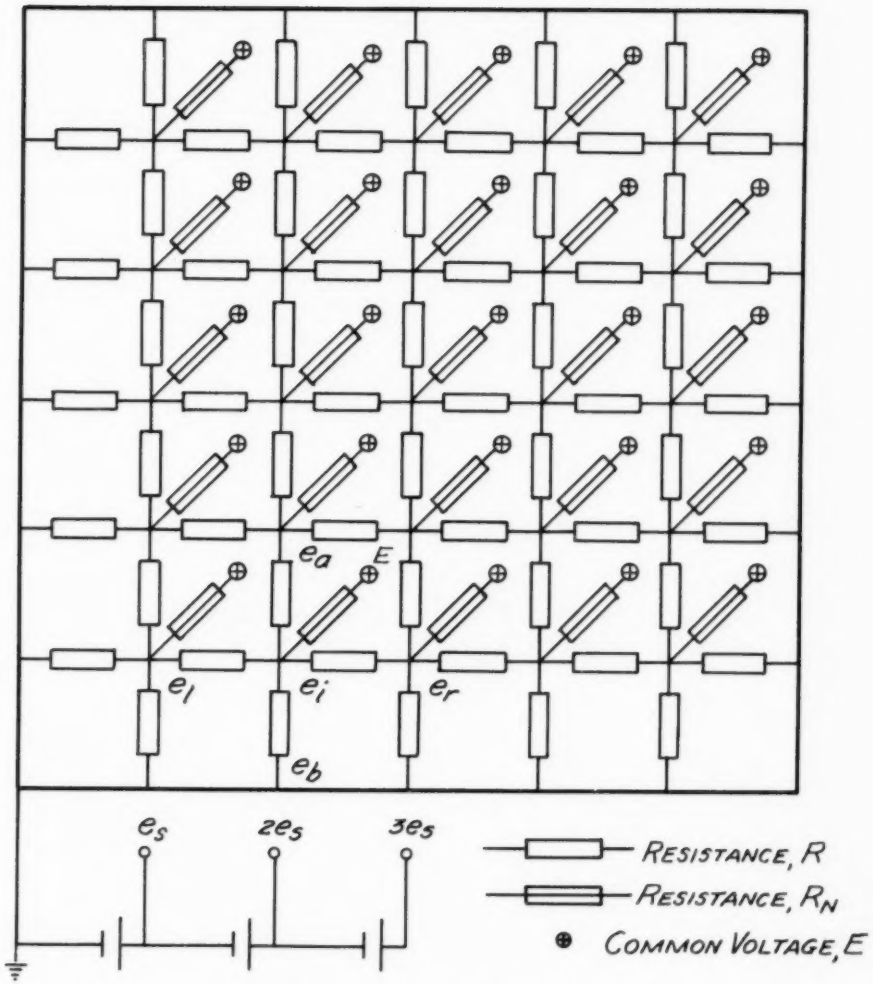

Fig. 3 Resistor Network

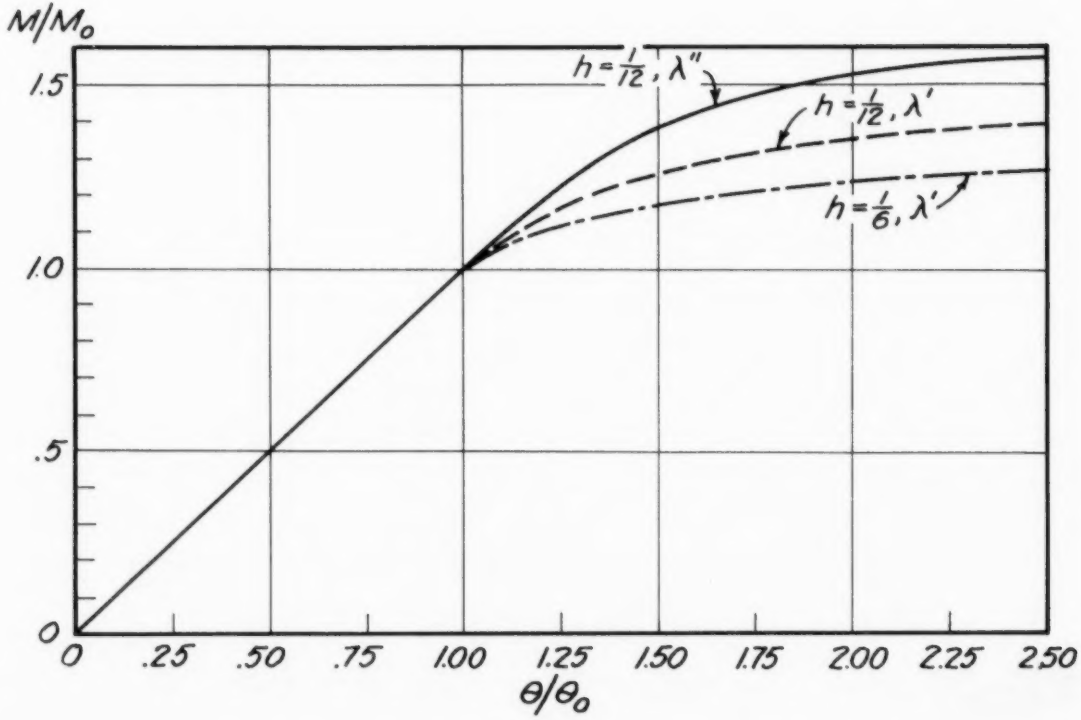

Fig. 4 Elasto-Plastic Torsion, Torque vs. Angle of Twist